

NATURAL CONNECTIONS ON CONFORMAL RIEMANNIAN P -MANIFOLDS

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Abstract

The class \mathcal{W}_1 of conformal Riemannian P -manifolds is the largest class of Riemannian almost product manifolds, which is closed with respect to the group of the conformal transformations of the Riemannian metric. This class is an analogue of the class of conformal Kähler manifolds in almost Hermitian geometry. In the present work¹ we study the natural connections on the manifolds (M, P, g) from the class \mathcal{W}_1 , i.e. the linear connections preserving the almost product structure P and the Riemannian metric g . We find necessary and sufficient conditions the curvature tensor of such a connection to have similar properties like the ones of the Kähler tensor in Hermitian geometry. We determine the type of the manifolds admitting a natural connection with a parallel torsion.

Key words: Riemannian almost product manifold, Riemannian metric, almost product structure, linear connection, parallel torsion.

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1. INTRODUCTION

K. Yano initiated in [1] the study of Riemannian almost product manifolds. A Riemannian almost product manifold (M, P, g) is a differentiable manifold M with an almost product structure P and a Riemannian metric g such that $P^2x = x$ and $g(Px, Py) = g(x, y)$ for any tangent vectors x and y . A. M. Naveira gave in [2] a classification of these manifolds with respect to the covariant derivative ∇P , where ∇ is the Levi-Civita connection of g . This classification is very similar to the Gray-Hervella classification in [3] of almost Hermitian manifolds. In the paper [4] M. Staikova and K. Gribachev obtained a classification of the Riemannian almost product manifolds, for which $\text{tr}P = 0$. In this case the manifold is even-dimensional. The class \mathcal{W}_0 from the Staikova-Gribachev classification is determined by the condition $\nabla P = 0$. A manifold from this class is called a *Riemannian P -manifold*. The class \mathcal{W}_0 is an analogue of the class of Kähler manifolds in almost Hermitian geometry.

The geometry of a Riemannian almost product manifold (M, P, g) is a geometry of both structures g and P . There are important in this geometry the linear connections in respect of which the parallel transport determine an isomorphism of the tangent spaces with the structures g and P . This is valid if and only if these structures are parallel with respect to such a

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connection. In the general case on a Riemannian almost product manifold there are countless number linear connections regarding which g and P are parallel. Such connections are called *natural* in [5].

In the present paper we consider some problems of the geometry of the natural connections on the class \mathcal{W}_1 from the Staikova-Gribachev classification. This is the class of *conformal Riemannian P -manifolds* or shortly *\mathcal{W}_1 -manifolds*. The class \mathcal{W}_1 is an analogue of the class of conformal Kähler manifolds in almost Hermitian geometry.

The paper is organized as follows. In Sec. 2 we give necessary facts about Riemannian almost product manifolds, the class \mathcal{W}_1 and the natural connections on \mathcal{W}_1 . We recall the notion a *Riemannian P -tensor* on a Riemannian almost product manifold, which is an analogue of the notion of a Kähler tensor in Hermitian geometry. In Sec. 3 we obtain relations between the curvature tensors R and R' , the Ricci tensors ρ and ρ' and the scalar curvatures τ and τ' of the Levi-Civita connection ∇ and a natural connection ∇' on a \mathcal{W}_1 -manifold. In Sec. 4 we find some necessary and sufficient conditions the curvature tensor R' of ∇' to be a Riemannian P -tensor. The most important result in this section is the classification Theorem 4.3. In it there are separated two special natural connections D and \tilde{D} , whose average connection is the canonical connection, introduced in [5]. In Sec. 5 we find conditions for a natural connection with parallel torsion. The important result here is Theorem 5.3. It characterizes a \mathcal{W}_1 -manifold with a natural connection, which has a parallel torsion and a Riemannian P -tensor of curvature.

2. PRELIMINARIES

Let (M, P, g) be a *Riemannian almost product manifold*, i.e. a differentiable manifold M with a tensor field P of type $(1, 1)$ and a Riemannian metric g such that $P^2x = x$, $g(Px, Py) = g(x, y)$ for any x, y of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M . Further x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space T_pM at $p \in M$. The *associated metric* \tilde{g} of g is determined by $\tilde{g}(x, y) = g(x, Py)$.

In this work we consider manifolds (M, P, g) with $\text{tr}P = 0$. In this case M is an even-dimensional manifold. We assume that $\dim M = 2n$.

In [2] A.M. Naveira gives a classification of Riemannian almost product manifolds with respect to the tensor F of type $(0, 3)$, defined by $F(x, y, z) = g((\nabla_x P)y, z)$, where ∇ is the Levi-Civita connection of g . The tensor F has the properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z).$$

Using the Naveira classification, in [4] M. Staikova and K. Gribachev give a classification of Riemannian almost product manifolds (M, P, g) with $\text{tr}P = 0$. The basic classes of this classification are \mathcal{W}_1 , \mathcal{W}_2 and \mathcal{W}_3 . Their intersection is the class \mathcal{W}_0 of the *Riemannian P -manifolds*, determined by

the condition $F = 0$ (or equivalently $\nabla P = 0$) [6]. This class is an analogue of the class of Kähler manifolds in the geometry of almost Hermitian manifolds.

A Riemannian almost product manifold (M, P, g) is a *Riemannian product manifold* if it has a local product structure, i.e. the structure P is integrable. The Riemannian product manifolds form the class $\mathcal{W}_1 \oplus \mathcal{W}_2$ from the classification in [4]. This class is an analogue of the class of Hermitian manifolds in almost Hermitian geometry.

The class \mathcal{W}_1 from the Staikova-Gribachev classification contains the manifolds which are locally conformal equivalent to Riemannian P -manifolds. This class plays a similar role of the role of the class of the conformal Kähler manifolds in almost Hermitian geometry. We will say that a manifold from the class \mathcal{W}_1 is a \mathcal{W}_1 -manifold.

The characteristic condition for the class \mathcal{W}_1 is the following

$$(2.1) \quad F(x, y, z) = \frac{1}{2n} \{ g(x, y)\theta(z) - g(x, Py)\theta(Pz) + g(x, z)\theta(y) - g(x, Pz)\theta(Py) \},$$

where the associated 1-form θ is determined by $\theta(x) = g^{ij}F(e_i, e_j, x)$. Here g^{ij} will stand for the components of the inverse matrix of g with respect to a basis $\{e_i\}$ of $T_p M$ at $p \in M$. The 1-form θ is *closed*, i.e. $d\theta = 0$, if and only if $(\nabla_x \theta)y = (\nabla_y \theta)x$. Moreover, $\theta \circ P$ is a closed 1-form if and only if $(\nabla_x \theta)Py = (\nabla_y \theta)Px$.

In [4] it is proved that $\mathcal{W}_1 = \overline{\mathcal{W}}_3 \oplus \overline{\mathcal{W}}_6$, where $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ are the classes from the Naveira classification determined by the following conditions:

$$\begin{aligned} \overline{\mathcal{W}}_3 : \quad & F(A, B, \xi) = \frac{1}{n}g(A, B)\theta^v(\xi), \quad F(\xi, \eta, A) = 0, \\ \overline{\mathcal{W}}_6 : \quad & F(\xi, \eta, A) = \frac{1}{n}g(\xi, \eta)\theta^h(A), \quad F(A, B, \xi) = 0, \end{aligned}$$

where $A, B, \xi, \eta \in \mathfrak{X}(M)$, $PA = A$, $PB = B$, $P\xi = -\xi$, $P\eta = -\eta$, $\theta^v(x) = \frac{1}{2}(\theta(x) - \theta(Px))$, $\theta^h(x) = \frac{1}{2}(\theta(x) + \theta(Px))$. In the case when $\text{tr}P = 0$, the above conditions for $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ can be written for any x, y, z in the following form:

$$(2.2) \quad \overline{\mathcal{W}}_3 : \quad F(x, y, z) = \frac{1}{2n} \{ [g(x, y) + g(x, Py)]\theta(z) + [g(x, z) + g(x, Pz)]\theta(y) \}, \quad \theta(Px) = -\theta(x),$$

$$(2.3) \quad \overline{\mathcal{W}}_6 : \quad F(x, y, z) = \frac{1}{2n} \{ [g(x, y) - g(x, Py)]\theta(z) + [g(x, z) - g(x, Pz)]\theta(y) \}, \quad \theta(Px) = \theta(x).$$

In [4], a tensor L of type (0,4) with properties

$$(2.4) \quad L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

$$(2.5) \quad L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0$$

is called a *curvature-like tensor*. Such a tensor on a Riemannian almost product manifold (M, P, g) with the property

$$(2.6) \quad L(x, y, Pz, Pw) = L(x, y, z, w)$$

is called a *Riemannian P -tensor* in [7]. This notion is an analogue of the notion of a Kähler tensor in Hermitian geometry.

Let S be a $(0,2)$ -tensor on a Riemannian almost product manifold. In [4] it is proved that the tensor $\psi_1(S)(x, y, z, w) = g(y, z)S(x, w) - g(x, z)S(y, w) + S(y, z)g(x, w) - S(x, z)g(y, w)$ is curvature-like if and only if $S(x, y) = S(y, x)$, and the tensor $\psi_2(S)(x, y, z, w) = \psi_1(S)(x, y, Pz, Pw)$ is curvature-like if and only if $S(x, Py) = S(y, Px)$. Obviously $\psi_2(S)(x, y, Pz, Pw) = \psi_1(S)(x, y, z, w)$. The tensors $\pi_1 = \frac{1}{2}\psi_1(g)$, $\pi_2 = \frac{1}{2}\psi_2(g)$, $\pi_3 = \psi_1(\tilde{g}) = \psi_2(\tilde{g})$ are curvature-like, and the tensors $\pi_1 + \pi_2$, π_3 are Riemannian P -tensors.

The linear connections in our investigations have a torsion. Let ∇' be a linear connection with a tensor Q of the transformation $\nabla \rightarrow \nabla'$ and a torsion T , i.e. $\nabla'_x y = \nabla_x y + Q(x, y)$, $T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]$. The corresponding $(0,3)$ -tensors are defined by $Q(x, y, z) = g(Q(x, y), z)$, $T(x, y, z) = g(T(x, y), z)$. The symmetry of ∇ implies $T(x, y) = -T(y, x) = Q(x, y) - Q(y, x)$.

A linear connection ∇' on a Riemannian almost product manifold (M, P, g) is called a *natural connection* if $\nabla'P = \nabla'g = 0$. If ∇' is a linear connection on (M, P, g) , then it is a natural connection if and only if [5]:

$$(2.7) \quad F(x, y, z) = Q(x, y, Pz) - Q(x, Py, z), \quad Q(x, y, z) = -Q(x, z, y).$$

The curvature tensor R of ∇ is determined by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]}z$ and the corresponding tensor of type $(0,4)$ is defined as follows $R(x, y, z, w) = g(R(x, y)z, w)$. We denote the Ricci tensor and the scalar curvature for ∇ by ρ and τ , respectively, i.e. $\rho(y, z) = g^{ij}R(e_i, y, z, e_j)$ and $\tau = g^{ij}\rho(e_i, e_j)$. Analogously there are defined the curvature tensor R' the Ricci tensor ρ' and the scalar curvature τ' for any connection ∇' .

Further ∇' will stand for a natural connection on a Riemannian almost product manifold (M, P, g) . Then it is valid the identity [8]

$$(2.8) \quad \begin{aligned} R(x, y, z, w) &= R'(x, y, z, w) - Q(T(x, y), z, w) - (\nabla'_x Q)(y, z, w) \\ &\quad + (\nabla'_y Q)(x, z, w) + g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)). \end{aligned}$$

In [9] it is established that the natural connections ∇' on a \mathcal{W}_1 -manifold (M, P, g) form a 2-parametric family, where the torsion T of ∇' on (M, P, g) is determined by

$$(2.9) \quad \begin{aligned} T(x, y, z) &= \frac{1}{2n} \{g(y, z)\theta(Px) - g(x, z)\theta(Py)\} \\ &\quad + \lambda \{g(y, z)\theta(x) - g(x, z)\theta(y) + g(y, Pz)\theta(Px) - g(x, Pz)\theta(Py)\} \\ &\quad + \mu \{g(y, Pz)\theta(x) - g(x, Pz)\theta(y) + g(y, z)\theta(Px) - g(x, z)\theta(Py)\}. \end{aligned}$$

3. CURVATURE PROPERTIES OF NATURAL CONNECTIONS ON \mathcal{W}_1 -MANIFOLDS

According to [10], for the torsion T of ∇' and the tensor Q of the transformation $\nabla \rightarrow \nabla'$ it is valid $2Q(x, y, z) = T(x, y, z) - T(y, z, x) + T(z, x, y)$.

Then from (2.9) for a \mathcal{W}_1 -manifold (M, P, g) holds

$$(3.1) \quad Q(x, y, z) = T(z, y, x).$$

Having in mind (3.1), the equality (2.9) implies

$$(3.2) \quad \begin{aligned} Q(y, z, w) &= g(y, z) \left\{ \lambda \theta(w) + \left(\mu + \frac{1}{2n} \right) \theta(Pw) \right\} \\ &\quad - g(y, w) \left\{ \lambda \theta(z) + \left(\mu + \frac{1}{2n} \right) \theta(Pz) \right\} \\ &\quad + g(y, Pz) \left\{ \lambda \theta(Pw) + \mu \theta(w) \right\} - g(y, Pw) \left\{ \lambda \theta(Pz) + \mu \theta(z) \right\}, \end{aligned}$$

$$(3.3) \quad \begin{aligned} Q(y, z) &= g(y, Pz) \left\{ \lambda P\Omega + \mu \Omega \right\} y - \left\{ \lambda \theta(z) + \left(\mu + \frac{1}{2n} \right) \theta(Pz) \right\} \\ &\quad + g(y, z) \left\{ \lambda \Omega + \left(\mu + \frac{1}{2n} \right) P\Omega \right\} - \left\{ \lambda \theta(Pz) + \mu \theta(z) \right\} Py, \end{aligned}$$

where Ω is determined by $g(\Omega, x) = \theta(x)$. According to (3.3), the connection ∇' , determined by the pair of parameters (λ, μ) has the form

$$(3.4) \quad \begin{aligned} \nabla'_x y &= \nabla_x y + g(x, Py) \left\{ \lambda P\Omega + \mu \Omega \right\} - \left\{ \lambda \theta(Py) + \mu \theta(y) \right\} Px \\ &\quad + g(x, y) \left\{ \lambda \Omega + \left(\mu + \frac{1}{2n} \right) P\Omega \right\} - \left\{ \lambda \theta(y) + \left(\mu + \frac{1}{2n} \right) \theta(Py) \right\} x. \end{aligned}$$

We denote $V(x, w) = \lambda (\nabla'_x \theta) Pw + \mu (\nabla'_x \theta) w$, $U(x, w) = \lambda (\nabla'_x \theta) w + \left(\mu + \frac{1}{2n} \right) (\nabla'_x \theta) Pw$. Since $\nabla' P = \nabla' g = \nabla' \tilde{g} = 0$, from (3.2) it follows

$$(3.5) \quad \begin{aligned} (\nabla'_x Q)(y, z, w) &= g(y, z) U(x, w) - g(y, w) U(x, z) \\ &\quad + g(y, Pz) V(x, w) - g(y, Pw) V(x, z). \end{aligned}$$

From (2.7) and (3.1) it follows $Q(T(x, y), z, w) = -g(T(x, y), T(z, w))$ which together with (3.3) and (3.5) we apply in (2.8). By appropriate calculations and using the notations

$$(3.6) \quad p = \lambda \Omega + \left(\mu + \frac{1}{2n} \right) P\Omega, \quad q = \lambda P\Omega + \mu \Omega,$$

$$(3.7) \quad S'(y, z) = U(y, z) - \frac{1}{2n} \left\{ \lambda \theta(y) \theta(Pz) + \mu \theta(y) \theta(z) \right\},$$

$$(3.8) \quad S''(y, z) = V(y, Pz) + \frac{1}{2n} \left\{ \lambda \theta(Py) \theta(Pz) + \mu \theta(Py) \theta(z) \right\},$$

we obtain the following

Theorem 3.1. *The following relation is valid*

$$(3.9) \quad R = R' - g(p, p) \pi_1 - g(q, q) \pi_2 - g(p, q) \pi_3 - \psi_1(S') - \psi_2(S'').$$

By the traces of the tensors in (3.9) we obtain the following

Corollary 3.2. *The following relations are valid*

$$\begin{aligned} \rho(y, z) &= \rho'(y, z) - \left\{ (2n-1)g(p, p) + \text{tr} S' - g(q, q) \right\} g(y, z) \\ &\quad - \text{tr} \tilde{S} g(y, Pz) - 2(n-1)S'(y, z) + S''(y, z) + S''(Py, Pz), \\ \tau &= \tau' - 2n(2n-1)g(p, p) + 2ng(q, q) - 2n(2n-1)\text{tr} S' + 2\text{tr} S'', \end{aligned}$$

where $\tilde{S}(y, z) = S(y, Pz)$.

4. NATURAL CONNECTIONS ON \mathcal{W}_1 -MANIFOLDS WITH RIEMANNIAN P -TENSOR OF CURVATURE

Because of $\nabla'g = \nabla'P = 0$, the properties (2.4) and (2.6) are valid for R' . Therefore R' is a Riemannian P -tensor if and only if R' satisfies (2.5). This property is valid for the curvature-like tensors R , π_1 , π_2 and π_3 . Then, according to (3.9), R' is a Riemannian P -tensor if and only if $\psi_1(S')$ and $\psi_2(S'')$ are curvature-like, i.e. $S'(y, z) = S'(z, y)$ and $S''(y, Pz) = S''(z, Py)$. In other words, using (3.7) and (3.8), it is valid the following

Proposition 4.1. *The curvature tensor of ∇' , determined by (3.4), is a Riemannian P -tensor if and only if the following conditions are valid:*

$$(4.1) \quad \begin{aligned} U(y, z) - U(z, y) + \frac{\lambda}{2n} \{ \theta(Py)\theta(z) - \theta(y)\theta(Pz) \} &= 0, \\ V(y, z) - V(z, y) + \frac{\mu}{2n} \{ \theta(Py)\theta(z) - \theta(y)\theta(Pz) \} &= 0. \end{aligned}$$

According to (3.3), there are valid the equalities

$$(4.2) \quad \begin{aligned} (\nabla'_y \theta) z - (\nabla'_z \theta) y &= (\nabla_y \theta) z - (\nabla_z \theta) y \\ &\quad - \frac{1}{2n} \{ \theta(Py)\theta(z) - \theta(y)\theta(Pz) \}, \\ (\nabla'_y \theta) Pz - (\nabla'_z \theta) Py &= (\nabla_y \theta) Pz - (\nabla_z \theta) Py \end{aligned}$$

and then Proposition 4.1 implies the following

Proposition 4.2. *The curvature tensor of ∇' , determined by (3.4), is a Riemannian P -tensor if and only if the following conditions are valid:*

$$(4.3) \quad \begin{aligned} \lambda \{ (\nabla_y \theta) z - (\nabla_z \theta) y \} + \left(\mu + \frac{1}{2n} \right) \{ (\nabla_y \theta) Pz - (\nabla_z \theta) Py \} &= 0, \\ \mu \{ (\nabla_y \theta) z - (\nabla_z \theta) y \} + \lambda \{ (\nabla_y \theta) Pz - (\nabla_z \theta) Py \} &= 0. \end{aligned}$$

The conditions (4.3) form a homogeneous linear system for $x_1 = (\nabla_y \theta) z - (\nabla_z \theta) y$, $x_2 = (\nabla_y \theta) Pz - (\nabla_z \theta) Py$ with a determinant $\Delta = \lambda^2 - \mu^2 - \frac{\mu}{2n}$.

Case I: $\Delta = 0$. Then (4.3) has a nonzero solution (x_1, x_2) and for λ and μ are possible the following three subcases: a) $\lambda = \mu = 0$; b) $\lambda = 0$, $\mu = -\frac{1}{2n}$; c) $\lambda \neq 0$, $\lambda^2 - \mu^2 - \frac{\mu}{2n} = 0$.

Let us consider the subcase a). If R' is a Riemannian P -tensor then the first equation of (4.3) implies that $\theta \circ P$ is closed. Since (4.3) has a nonzero solution then θ is not closed. Vice versa, if $\theta \circ P$ is closed, then, according to [8], R' is a Riemannian P -tensor.

Let us consider the subcase b). If R' is a Riemannian P -tensor then the second equation of (4.3) implies that θ is a closed. Since (4.3) has a nonzero solution then $\theta \circ P$ is not closed. Vice versa, if θ is closed, then the first equation of (4.2) implies $(\nabla'_y \theta) z - (\nabla'_z \theta) y = \frac{1}{2n} \{ \theta(y)\theta(Pz) - \theta(Py)\theta(z) \}$ and since $\lambda = 0$ the second equality of (4.3) holds. Obviously, the first equation of (4.3) is valid, too. Then, according to Proposition 4.1, R' is a Riemannian P -tensor.

Let us consider the subcase c). Thus we have also $\mu \neq 0$. If R' is a Riemannian P -tensor, the second equation of (4.3) yields that θ and $\theta \circ P$

are not closed, since the system (4.3) has a nonzero solution. In this subcase \mathcal{W}_1 -manifold (M, P, g) is not belong to the classes $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$. Indeed, in the opposite case we have $\theta(Pz) = \varepsilon\theta(z)$ ($\varepsilon = \pm 1$) and therefore the following equality is valid

$$(\nabla_y \theta) Pz - (\nabla_z \theta) Py = \varepsilon \{(\nabla_y \theta) z - (\nabla_z \theta) y\} - F(y, z, \Omega) + F(z, y, \Omega).$$

Since (2.2) and (2.3), it follows $F(y, z, \Omega) = \frac{1}{2n} \{g(y, z) - \varepsilon g(y, Pz)\} \theta(\Omega)$. Using the last two equalities, the system (4.3) gets the form

$$(4.4) \quad \begin{aligned} (\lambda + \varepsilon(\mu + \frac{1}{2n})) \{(\nabla_y \theta) z - (\nabla_z \theta) y\} &= 0, \\ (\mu + \varepsilon\lambda) \{(\nabla_y \theta) z - (\nabla_z \theta) y\} &= 0. \end{aligned}$$

Because of $\lambda \neq 0$, $\lambda^2 - \mu^2 - \frac{\mu}{2n} = 0$ in the considered subcase, we have $\lambda + \varepsilon(\mu + \frac{1}{2n}) \neq 0$, $\mu + \varepsilon\lambda \neq 0$. Then (4.4) implies that θ is closed, which is a contradiction. Therefore $(M, P, g) \notin \overline{\mathcal{W}}_3$ and $(M, P, g) \notin \overline{\mathcal{W}}_6$.

Case II: $\Delta \neq 0$. If R' is a Riemannian P -tensor then the conditions (4.3) are satisfied. Since the homogeneous system has the zero solution $x_1 = x_2 = 0$ in this case then θ and $\theta \circ P$ are closed. Vice versa, if θ and $\theta \circ P$ are closed, then (4.2) implies the equalities $(\nabla'_y \theta) Pz - (\nabla'_z \theta) Py = 0$, $(\nabla'_y \theta) z - (\nabla'_z \theta) y = \frac{1}{2n} \{\theta(y)\theta(Pz) - \theta(Py)\theta(z)\}$ and thus conditions (4.1) hold. Then, according to Proposition 4.1, R' is a Riemannian P -tensor.

The obtained results we summarize in the following classification theorem for the natural connections on a \mathcal{W}_1 -manifold.

Theorem 4.3. *Let R' be the curvature tensor of a natural connection ∇' determined by (3.4) on a \mathcal{W}_1 -manifold (M, P, g) . Then the all possible cases are as follows:*

- i) *If ∇' is the connection D determined by $\lambda = \mu = 0$, then R' is a Riemannian P -tensor if and only if the 1-form θ is not closed and the 1-form $\theta \circ P$ is closed;*
- ii) *If ∇' is the connection \tilde{D} determined by $\lambda = 0$, $\mu = -\frac{1}{2n}$, then R' is a Riemannian P -tensor if and only if the 1-form θ is closed and the 1-form $\theta \circ P$ is not closed;*
- iii) *If ∇' is a connection for which $\lambda^2 - \mu^2 - \frac{\mu}{2n} \neq 0$, then R' is a Riemannian P -tensor if and only if the 1-forms θ and $\theta \circ P$ are closed;*
- iv) *If ∇' is a connection for which $\lambda \neq 0$, $\lambda^2 - \mu^2 - \frac{\mu}{2n} = 0$ and R' is a Riemannian P -tensor then the 1-forms θ and $\theta \circ P$ are not closed. In this case $(M, P, g) \notin \overline{\mathcal{W}}_3$, $(M, P, g) \notin \overline{\mathcal{W}}_6$.*

Let us remark that the proposition i) of Theorem 4.3 is proved in [8], where the connection D is investigated.

In [5], it is introduced a natural connection ∇^c on an arbitrary Riemannian almost product manifold (M, P, g) . This connection, called *canonical*, is an analogue of the Hermitian connection in Hermitian geometry. Analogues of this connection are considered on manifolds, corresponding to the class \mathcal{W}_1 ,

in the geometry of almost complex manifolds with Norden metric ([11]) and almost contact manifolds with B-metric ([12],[13]). In [9], it is noted that ∇^c on a Riemannian almost product \mathcal{W}_1 -manifold is determined by (3.4) for $\lambda = 0$, $\mu = -\frac{1}{4n}$. Therefore, for ∇^c it is valid the condition $\lambda^2 - \mu^2 - \frac{\mu}{2n} \neq 0$ and then the statement iii) of Theorem 4.3 holds.

Having in mind the values of λ and μ determining the connections ∇^c , D and \tilde{D} , we obtain the following

Proposition 4.4. *The canonical connection ∇^c is the average connection of D and \tilde{D} , i.e. $\nabla^c = \frac{1}{2}(D + \tilde{D})$.*

5. NATURAL CONNECTION ON A \mathcal{W}_1 -MANIFOLD WITH PARALLEL TORSION

Calculating from (2.9) the covariant derivative $\nabla' T$ and applying $\nabla' g = \nabla' \tilde{g} = 0$, we obtain the following

Proposition 5.1. *A natural connection ∇' has a parallel torsion if and only if the 1-form θ is parallel with respect to ∇' .*

By (4.2), (2.2) and (2.3), from Proposition 5.1 we obtain the following

Corollary 5.2. *If a \mathcal{W}_1 -manifold (M, P, g) admits a natural connection with parallel torsion then the 1-form $\theta \circ P$ is closed. If (M, P, g) belongs to the class $\overline{\mathcal{W}}_3$ or the class $\overline{\mathcal{W}}_6$ then the 1-form θ is closed, too.*

Let ∇' be a natural connection with parallel torsion and Riemannian P -tensor of curvature. Then from Proposition 5.1 we have $U = V = 0$. In this case Proposition 4.1 implies $\lambda W = \mu W = 0$, where $W(y, z) = \theta(Py)\theta(z) - \theta(y)\theta(Pz)$. If $W = 0$ then we obtain $\theta(Py) = \pm\theta(y)$, i.e. $(M, P, g) \in \overline{\mathcal{W}}_3$ or $(M, P, g) \in \overline{\mathcal{W}}_6$. Thus, we get the following

Theorem 5.3. *Let a \mathcal{W}_1 -manifold (M, P, g) admits a natural connection ∇' with parallel torsion and Riemannian P -tensor of curvature. Then the all possible cases are as follows:*

- i) $\lambda = \mu = 0$, $W \neq 0$, i.e. $\nabla' = D$, $(M, P, g) \notin \overline{\mathcal{W}}_3$, $(M, P, g) \notin \overline{\mathcal{W}}_6$;
- ii) $\lambda = \mu = 0$, $W = 0$, i.e. $\nabla' = D$, $(M, P, g) \in \overline{\mathcal{W}}_3$ or $(M, P, g) \in \overline{\mathcal{W}}_6$;
- iii) $\lambda = 0$, $\mu \neq 0$, $W = 0$, i.e. ∇' belongs to the 1-parametric family determined by (3.4) for $\lambda = 0$, $(M, P, g) \in \overline{\mathcal{W}}_3$, $(M, P, g) \in \overline{\mathcal{W}}_6$;
- iv) $\lambda \neq 0$, $\mu = 0$, $W = 0$, i.e. ∇' belongs to the 1-parametric family determined by (3.4) for $\mu = 0$, $(M, P, g) \in \overline{\mathcal{W}}_3$, $(M, P, g) \in \overline{\mathcal{W}}_6$.

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